Math 250

Chapter 6 : Applications of Integration

6.1 Velocity and Net Change
6.2 Regions between Curves
6.3 Volumes by Slicing (Disks/Washers)
6.4 Volumes by Shells

skip the remainder of Chapter 6 — part of Math 251

6.1 Velocity and Net Change

Objectives

1) Understand the Net Change Theorem
2) Use the concept of Future Value as it relates to the Net Change Theorem
3) Apply these concepts to rectilinear motion to
   - calculate displacement
   - calculate total distance traveled
   - find the future value of a position function
   - find the position function from the velocity
   - find the future value of a velocity function

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The Net Change Theorem: Is a restatement of the FTC with different notation, allowing for expanded uses.

FTC says $\int_a^b f(x)\,dx = F(b) - F(a)$ for $F'(x) = f(x)$, $F(x)$ an antiderivative of $f(x)$.

Net Change Theorem Says: $\int_a^b F'(x)\,dx = F(b) - F(a)$

"The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or net change, in that quantity on the interval $[a,b]$."

Also called "integrating a rate of change"

Another way to see this is by remembering that

F is an antiderivative of f

means

$F'(x) = f(x)$.

So $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$

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<u>Position function</u> $x(t)$ = location on x-axis at time t.

If $x(t) > 0$, particle is located to the right of the origin, at a positive x-coordinate.

If $x(t) < 0$, particle is located to the left of the origin, at a negative x-coordinate.

<u>Velocity Function</u> $v(t)$ = speed and direction of particle at time t.

$v(t) = x'(t)$

If $v(t) > 0$, particle is moving to the right.

If $v(t) < 0$, particle is moving to the left.

$v(t) = x'(t)$.

If we apply the Net Change Theorem with $F(x) = x(t)$ and $F'(x) = x'(t) = v(t)$,

we get $\displaystyle\int_a^b v(t)\,dt = x(b) - x(a)$

$x(b)$: position at end; $x(a)$: position at beginning

which is <u>displacement</u> on $[a,b]$.

Less Work
does not measure back + forth movement

If we want to know the total distance traveled by the particle, we cannot add negative $v(t)$ values in the integral.

<u>Total distance</u> traveled on $[a,b] = \displaystyle\int_a^b |v(t)|\,dt$.

More Work
measures every back + forth

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If we are considering a function of time, say a quantity $Q(t)$

Net Change says: $\int_a^b Q'(t)\,dt = Q(b) - Q(a)$

If we consider the initial time $t=a$ to be $0 \rightarrow a=0$

$$\int_0^b Q'(t) = Q(b) - Q(0).$$

so $Q(b)$ is the quantity at a future time $t=b$.
and we can isolate it to get

$$\boxed{Q(0) + \int_0^b Q'(t) = Q(b) \leftarrow \frac{\text{Future Value}}{\text{of } Q(t)}}$$

If we apply this to

- velocity and acceleration:

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

- position and velocity:

$$s(t) = s(0) + \int_0^t v(x)\,dx$$

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① A cat paces back and forth along a 40-foot length of fence with velocity function $v(t) = 20\cos(t)$ and initial position in the middle of the fence, $s(0) = 20$ with length in feet and time in minutes.

a) Find the position of the cat.

b) Graph the position $0 \le t \le 9\pi$.

c) Approximately how long does the cat pace?

d) At what times is the cat in the middle of the fence?

e) Find the displacement of the cat $0 \le t \le \frac{5\pi}{4}$

f) Find the total distance traveled by the cat $0 \le t \le \frac{5\pi}{4}$

a) position $s(t) = s(0) + \int_0^t v(x)\,dx$

Future value formula A.K.A. Solve the differential equation
$s'(t) = 20\cos(t)$
$s(0) = 20$

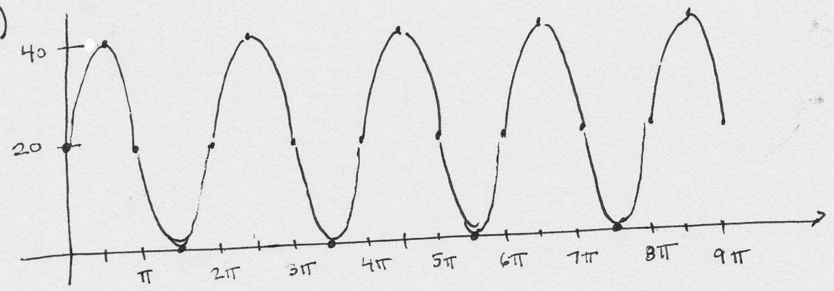
$$s(t) = 20 + \int_0^t 20\cos(x)\,dx$$

$$= 20 + 20\left[\sin x\right]\Big|_0^t$$

$$= 20 + 20\sin t - 20\sin 0$$

$$\boxed{s(t) = 20 + 20\sin t}$$

b)



c) $9\pi \approx \boxed{28.3 \text{ min}}$

d) middle of fence $= s(t) = 20$

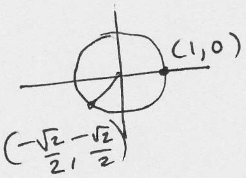
$\boxed{t = 0, \pi, 2\pi \ldots n\pi \quad n = 1 \text{ to } 9}$

e) displacement $= \int_0^{\frac{5\pi}{4}} v(t)\,dt$

$$= \int_0^{\frac{5\pi}{4}} 20\cos(t)\,dt$$

$$= 20\sin(t)\Big|_0^{\frac{5\pi}{4}}$$

$$= 20\sin\left(\frac{5\pi}{4}\right) - 20\sin(0)$$



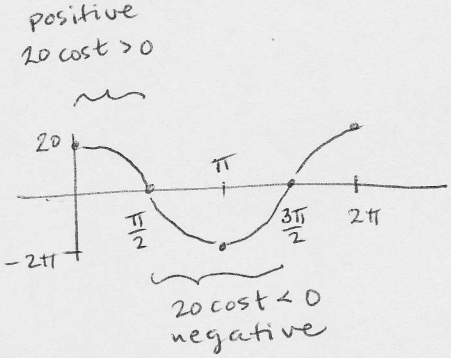
$$= 20\left(-\frac{\sqrt{2}}{2}\right) - 20(0)$$

$$= \boxed{-10\sqrt{2}\text{ ft}} \text{ (to left)}$$

$$\approx \boxed{-14.14\text{ ft}}$$

The cat is ≈ 14 ft to the left of center or at position $20 - 10\sqrt{2} \approx 5.86$

f) Total distance $= \int_0^{\frac{5\pi}{4}} |v(t)|\,dt$



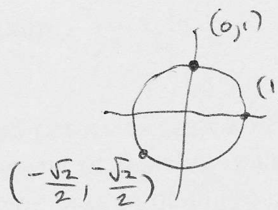
find where is function is negative & take its opposite

$$= \int_0^{\frac{5\pi}{4}} |20\cos t|\,dt$$

$$= \int_0^{\frac{\pi}{2}} 20\cos t\,dt + \int_{\pi/2}^{5\pi/4} -20\cos t\,dt$$

$$= 20\int_0^{\pi/2} \cos t\,dt - 20\int_{\pi/2}^{5\pi/4} \cos t\,dt$$

$$= 20\sin t\Big|_0^{\pi/2} - 20\sin t\Big|_{\pi/2}^{5\pi/4}$$



$$= \left(20\sin\frac{\pi}{2} - 20\sin 0\right) - \left(20\sin\frac{5\pi}{4} - 20\sin\frac{\pi}{2}\right)$$

$$= (20(1) - 2(0)) - \left(20\cdot\left(-\frac{\sqrt{2}}{2}\right) - 20\cdot(1)\right)$$

$$= 20 - 0 - (-10\sqrt{2} - 20)$$

$$= 20 - 0 + 10\sqrt{2} + 20$$

$$= \boxed{40 + 10\sqrt{2}\text{ ft}}$$

$$\approx \boxed{54.14\text{ ft}}$$

all of the cat's movement is counted.

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× 1. A particle is moving along the x-axis with position function at time t given by $x(t) = (t-1)(t-3)^2$, for $0 \le t \le 5$.

a) Find the displacement in 5 units of time

b) Find the total distance traveled in 5 units of time.

a) Displacement $= \int_0^5 x'(t)\,dt = x(5) - x(0)$

$= 16 - (-9)$

$= \boxed{25}$

$x(5) = (5-1)(5-3)^2$

$= 4(2)^2$

$= 16$

$x(0) = (0-1)(0-3)^2$

$= -1(9)$

$= -9$

b) Total distance $= \int_0^5 |x'(t)|\,dt$.

step 1: find $v(t) = x'(t)$.

step 2: find where $x'(t) < 0$

step 3: split up $\int_0^5$ into integrals of positive quantities.

where is $v(t)$ positive

and where is $v(t)$ negative?

cont →

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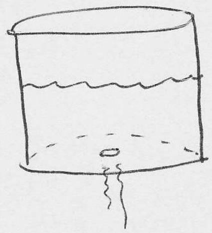
(3) According to Torricelli's Law, a cylindrical tank with a round drain empties with rate of change

$$V'(t) = \frac{\pi r^4 g t}{R^2} - \pi r^2 \sqrt{2gH}$$

for $t \geq 0$, time since drain opened



R = radius of cylinder
H = height of cylinder
r = radius of drain
g = acceleration due to gravity, (expressed as absolute value)

a) Find the function for the rate of change of the volume for a tank with $r = 5$ cm, $R = 50$ cm, $H = 100$ cm, and $g = 980$ cm/sec^2.

b) Find total capacity of the tank.

c) Find the volume function $V(t)$ if the tank is full at time 0.

d) Sketch graph of $V(t)$.

a) Plug in $r=5$, $R=50$, $H=100$, $g=980$:

$$V'(t) = \frac{\pi (5)^4 (980)\, t}{(50)^2} - \pi (5)^2 \sqrt{2(980)100}$$

$$= 245\pi t - 25\pi \sqrt{196000}$$

$14^2 = 196$
$10^2 = 100$

$$= 245\pi t - 25(10)(14)\sqrt{10}\,\pi$$

$$\boxed{V'(t) = 245\pi t - 3500\pi\sqrt{10}}$$

$$V'(t) = 35\pi (7t - 100\pi\sqrt{10})$$ factored

b) $V = \pi R^2 H = \pi (50)^2 (100) = \boxed{250{,}000\pi \text{ cm}^3}$

c) $$V(t) = V(0) + \int_0^t V'(x)\,dx$$

Future value formula with dummy variable

$$V(t) = 250{,}000\pi + \int_0^t 245\pi x - 3500\pi\sqrt{10}\; dx$$

$$= 250{,}000\pi + 245\pi \int_0^t x\,dx - 3500\pi\sqrt{10}\int_0^t 1\,dx$$

$$= 250{,}000\pi + 245\pi \cdot \left(\frac{x^2}{2}\right)\Big|_0^t - 3500\pi\sqrt{10}\,(x)\Big|_0^t$$

$$= 250{,}000\pi + \frac{245\pi}{2}t^2 - 3500\pi\sqrt{10}\, t$$

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$$\boxed{V(t) = \frac{245}{2}\pi t^2 - 3500\pi\sqrt{10}\, t + 250000\pi}$$

d) $V(t)$ is a parabola $V(t) = at^2 + bt + c$ $\quad a > 0$

$$\text{vertex} = \frac{-b}{2a} = \frac{-(-3500\pi\sqrt{10})}{2\left(\frac{245}{2}\pi\right)}$$

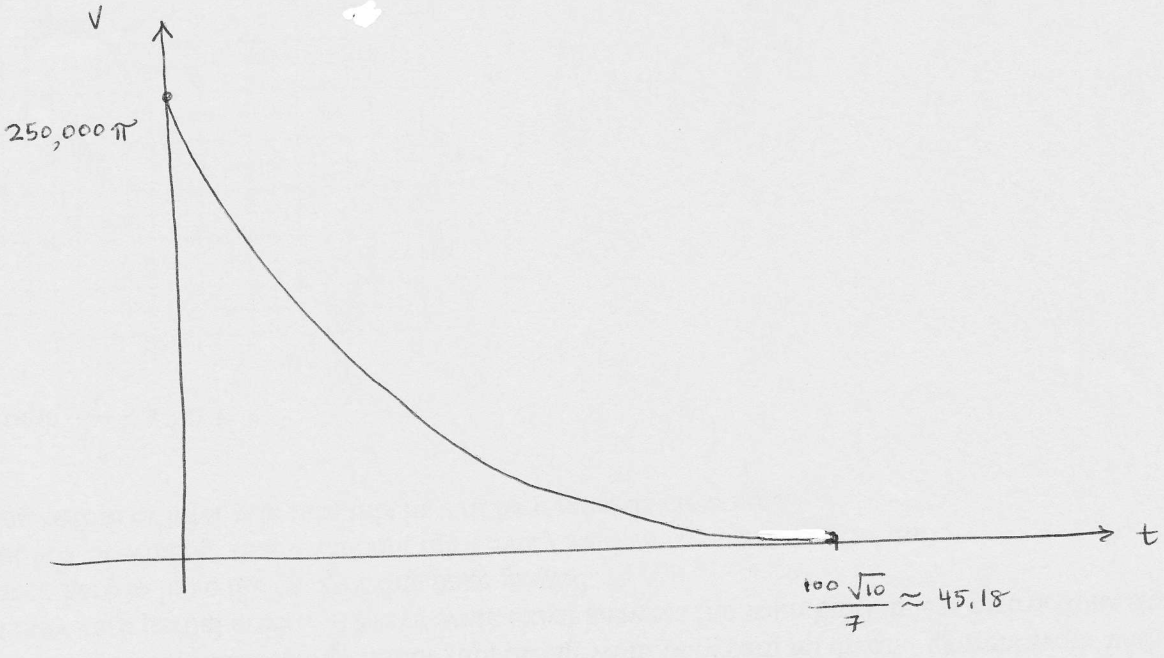
$$= \frac{3500\pi\sqrt{10}}{245\pi}$$

$$= \frac{3500\sqrt{10}}{245}$$

$$t = \frac{100}{7}\sqrt{10} \approx 45.18 \text{ sec}$$

$$V\left(\frac{100}{7}\sqrt{10}\right) = \frac{245}{2}\pi\left(\frac{10}{7}\sqrt{10}\right)^2 - 3500\pi\sqrt{10}\left(\frac{10}{7}\sqrt{10}\right) + 250{,}000\pi$$

$$= 0$$



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(4) A population of foxes is observed over 10 years. The growth rate of the fox population is

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

with initial population $P(0) = 35$.

a) Find the population $P(t)$ for any $t \geq 0$.

b) Find the population after 15 years & after 35 years.

a) $P(t) = P(0) + \int_0^t P'(x)\,dx$ Future value formula

$$= 35 + \int_0^t 5 + 10\sin\left(\frac{\pi \cancel{t} x}{5}\right)dx$$

$$= 35 + 5\int_0^t dx + 10\int_0^t \sin\left(\frac{\pi x}{5}\right)dx$$

$$= 35 + 5x\Big|_0^t + \frac{10 \cdot 5}{\pi}\int_0^{\frac{\pi}{5}t} \sin u\,du$$

$u = \frac{\pi x}{5}$ $du = \frac{\pi}{5}dx$

$u_1 = \frac{\pi}{5}(0) = 0$

$u_2 = \frac{\pi}{5}t$

$$= 35 + 5t + \frac{50}{\pi}\left[-\cos u\right]\Big|_0^{\frac{\pi}{5}t}$$

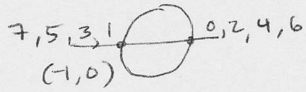
$$= 35 + 5t - \frac{50}{\pi}\left(\cos\left(\frac{\pi}{5}t\right) - \cos(0)\right)$$

$$= 35 + 5t - \frac{50}{\pi}\cos\left(\frac{\pi}{5}t\right) + \frac{50}{\pi}$$

$$\boxed{P(t) = 35 + \frac{50}{\pi} + 5t - \frac{50}{\pi}\cos\left(\frac{\pi}{5}t\right)}$$

b) $P(15) = 35 + \frac{50}{\pi} + 5(15) - \frac{50}{\pi}\cos\left(\frac{\pi}{5}\cdot 15\right)$

$$= 35 + \frac{50}{\pi} + 75 - \frac{50}{\pi}\cos(3\pi)$$



$$= 110 + \frac{50}{\pi} - \frac{50}{\pi}(-1)$$

$$= 110 + \frac{100}{\pi} \approx 141.8 \rightarrow \boxed{142 \text{ foxes}}$$

c) $P(35) = 35 + \frac{50}{\pi} + 5(35) - \frac{50}{\pi}\cos\left(\frac{\pi}{5}\cdot 35\right)$

$$= 210 + \frac{100}{\pi} \approx 241.8 \rightarrow \boxed{242 \text{ foxes}}$$